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# Naked singularities in dilatonic domain wall space-times

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## Abstract

We investigate gravitational effects of extreme, non-extreme and ultra-extreme domain walls in the presence of a dilaton field  $\phi$ . The dilaton is a scalar field without self-interaction that couples to the matter potential that is responsible for the formation of the wall. Motivated by superstring and supergravity theories, we consider both an exponential dilaton coupling (parametrized with the coupling constant  $\alpha$ ) and the case where the coupling is self-dual, *i.e.* it has an extremum for a finite value of  $\phi$ . For an exponential dilaton coupling ( $e^{2\sqrt{\alpha}\phi}$ ), extreme walls (which are static planar configurations with surface energy density  $\sigma_{\text{ext}}$  saturating the corresponding Bogomol'nyi bound) have a *naked* (planar) singularity outside the wall for  $\alpha > 1$ , while for  $\alpha \leq 1$  the singularity is null. On the other hand, non-extreme walls (bubbles with two insides and  $\sigma_{\text{non}} > \sigma_{\text{ext}}$ ) and ultra-extreme walls (bubbles of false vacuum decay with  $\sigma_{\text{ultra}} < \sigma_{\text{ext}}$ ) *always* have naked singularities. There are solutions with self-dual couplings, which reduce to singularity-free vacuum domain wall space-times. However, only non- and ultra-extreme walls of such a type are dynamically stable.

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## I. INTRODUCTION

Domain walls are surfaces interpolating between regions of space-time with different expectation values of some matter field(s). A domain wall [1] is a vacuum-like hypersurface where the positive tension equals the mass density. In field theory, domain wall configurations are possible if the effective potential of the corresponding matter field(s) has more than one isolated minimum. If each side of the domain wall corresponds to a vacuum, that is, if on either side all the matter fields have *constant* expectation values, then all the local properties of each side are Lorentz-invariant and each side is a *vacuum*. Such domain walls are usually referred to as *vacuum domain walls*.

Vacuum domain walls can be classified according to the value of their surface energy density  $\sigma$ , compared to the energy densities of the vacua outside the wall [2,3]. The three types are: (1) extreme walls with  $\sigma = \sigma_{\text{ext}}$  are planar, static walls. In this case there is a perfect balance between the gravitational mass of the wall and that of the exterior vacua; (2) non-extreme walls with  $\sigma = \sigma_{\text{non}} > \sigma_{\text{ext}}$  corresponding to non-static bubbles with two centres and (3) ultra-extreme walls with  $\sigma = \sigma_{\text{ultra}} < \sigma_{\text{ext}}$  representing expanding bubbles of false vacuum decay.

The extreme domain wall solutions were found [4,5] in  $N = 1$  supergravity theory, representing walls interpolating between isolated supersymmetric vacua of the matter potential. They correspond to supersymmetric configurations saturating the corresponding Bogomol'nyi bound. There are three possible types [5] of these extreme vacuum domain walls. They are classified according to the nature of the field path in the superpotential or according to the type of the induced space-times. Type I walls interpolate between supersymmetric anti-de Sitter and Minkowski vacua. Type II and Type III walls interpolate between two supersymmetric anti-de Sitter vacua. For a Type II wall the superpotential  $W(T)$  passes through zero, and the conformal factor of the space-time metric decreases away from the wall on both sides. For Type III walls the superpotential does not pass through zero, and the metric conformal factor is a monotonously increasing function of  $z$ , where  $z$  maps the spatial direction perpendicular to the wall.

The global structure of the induced space-times of the extreme domain wall configurations has been studied in Refs. [6–8], and generalized to and compared with the non-extreme ones in Refs. [2,3,8]. Interestingly, the  $(t, z)$  space-time slice of the Type I extreme [non-extreme] walls exhibits the *same* global space-time structure as the one of the corresponding extreme [or non-extreme] charged black holes; however, now the time-like singularity of the charged black holes is replaced by another wall.

On the other hand, the existence of dilatons is a generic feature of unifying theories, including effective actions of superstring vacua, certain classes of supergravity theories as well as Kaluza-Klein theories. ‘Dilaton’ is here used as a generic name for a scalar field without self-interactions that couple to the matter sources, *i.e.* the potential of scalar matter fields as well as the kinetic energy of gauge fields, and thereby it modulates the overall strength of such interactions. In the low-energy effective action of superstring theory, the dilaton plays an essential rôle for the “scale-factor duality” [9], which has been taken as an indication of a “dual pre-big-bang” phase as a possible alternative to the initial singularity of the standard cosmological model [10]. The dilaton is believed to play a crucial role in dynamical supersymmetry breaking as well.

Moreover, in theories with dilaton field(s), topological defects in general, and black holes in particular, have a space-time structure that is drastically changed compared to the non-dilatonic ones. Namely, since the dilaton couples to the matter sources, *e.g.* the charge of the black hole, or it modulates the strength of the matter interactions, it in turn changes the nature of the space-time. In the past, charged dilatonic black holes have been studied extensively (for a review see Ref. [11] and references therein).

It is therefore of considerable interest to generalize another type of topological defects, *i.e.* the vacuum domain wall solutions [12–18,2,3], by including the dilaton, thus addressing the nature of space-time in the domain wall background with a varying dilaton field. Such configurations may be of specific interest in the study of domain walls in the early universe as they may arise in fundamental theories that include the dilaton, in particular in an effective theory from superstrings. In addition, the nature of ultra-extreme dilatonic domain walls, which describe false vacuum decay, is of importance in basic theories that contain one or more dilaton fields.

The first set of extreme dilatonic domain wall solutions [19,20] was found in  $N = 1$  supergravity theory coupled to a linear supermultiplet. There the dilaton  $\phi$  corresponds to the scalar component of the linear multiplet;  $\phi$  couples with an exponential coupling  $e^{2\sqrt{\alpha}\phi}$  to the potential of the matter scalar fields, and with a “complementary” exponential coupling  $e^{-2\phi/\sqrt{\alpha}}$  to the kinetic energy of the gauge fields. The coupling  $\alpha = 1$  is the one of the effective tree level action of the superstring. The extreme domain wall configurations for such actions correspond to static domain walls interpolating between supersymmetric minima of the matter potential. However, along with the dependence of the space-time metric on the coordinate distance from the wall,  $z$ , also the dilaton field now varies with  $z$ . In particular, the Type I walls, which interpolate between the supersymmetric Minkowski space-time, with matter superpotential satisfying  $W(T) = 0$  and a constant dilaton on the one side, and a new type of supersymmetric space-time where  $W(T) \neq 0$  and where the dilaton varies with  $z$  on the other. The space-time structure on the latter side crucially depends on the value of the coupling constant  $\alpha$ . For  $\alpha \leq 1$  there is a planar null singularity, while for  $\alpha > 1$  the singularity is *naked*. At the singularity both the dilaton field and the space-time curvature diverge.

In the  $(t, z)$  slice, the global space-time structure of the Type I extreme dilatonic walls with coupling  $\alpha$  turns out to be the same [20] as the  $(t, r)$  slice of the extreme charged dilatonic black holes with coupling  $1/\alpha$ . The complementarity between the global space-time structure of the extreme dilatonic domain walls with coupling  $\alpha$  and extreme charged dilatonic black holes with coupling  $1/\alpha$  can be traced back to the nature of the coupling  $e^{2\sqrt{\alpha}\phi}$  of the dilaton to the matter potential (the source for the wall) and the complementary coupling  $e^{-2\phi/\sqrt{\alpha}}$  of the dilaton to the gauge kinetic energy (the source of the charge of the black hole). The newly found complementarity ( $\alpha \leftrightarrow 1/\alpha$ ) between the extreme wall and extreme charged black hole solutions is a generalization of the one found [6] between extreme vacuum domain walls ( $\alpha = 0$ ) and ordinary extreme black holes ( $\alpha = \infty$ ). Interestingly, only for the  $N = 1$  supergravity with the coupling  $\alpha = 1$ , which corresponds to an effective tree level theory from superstrings, are both extreme dilatonic walls *and* extreme charged dilatonic black holes void of naked singularities.

In the present paper we further investigate dilatonic domain walls. Within  $N = 1$  supergravity we generalize the extreme dilatonic solutions to the case with an arbitrary

separable dilaton Kähler potential. We derive the corresponding Bogomol'nyi equations for the static, supersymmetric walls, and in the thin wall approximation also the energy density of such walls, which saturates the corresponding Bogomol'nyi bound. Since the form of the separable Kähler potential is kept arbitrary, the analysis applies also to the self-dual case, *i.e.* when the Kähler potential has an extremum for a finite dilaton value.

The major part of this paper involves a study of the space-time for non-extreme and ultra-extreme dilatonic domain walls in the thin wall approximation. The solutions are non-static bubbles. These solutions can be parametrized by a parameter  $\beta$  that measures deviation from extremality; that is to say,  $\beta = 0$  represents the corresponding extreme solution. Within  $N = 1$  supergravity such walls would correspond to solutions which interpolate between two isolated minima of the matter potential, where at least one of the isolated minima breaks supersymmetry. These walls are generalizations of the corresponding non- and ultra-extreme vacuum domain walls, but now, only numerical solutions have been obtained. For this reason, the analysis can be done only for walls for which the boundary conditions for the dilaton field and the metric can be specified uniquely at the wall surface. Nonetheless, such cases include the physically interesting example of Type I walls, which interpolate between Minkowski space-time with a constant dilaton value and a new type of space-time with varying dilaton, as well as reflection-symmetric (non-extreme) walls.

Interestingly, the space-time induced in the dilatonic domain wall backgrounds can be related to certain cosmological solutions by a complex coordinate transformation where  $z$  is replaced with a cosmic time coordinate and where the potential changes sign. The non-extremality parameter  $\beta$  then plays the rôle of a cosmological spatial curvature:  $\beta^2 \rightarrow -k$ . Thus, the nature of the dilatonic domain wall space-time solutions can be related to the corresponding cosmological solutions, and where it is possible, we draw the analogy.

We concentrate on the case with a general exponential dilaton coupling ( $e^{2\sqrt{\alpha}\phi}$ ) to the matter potential that creates the wall. It turns out that—unlike the extreme case—for any  $\alpha$  and any non-zero value of the non-extremality parameter  $\beta$ , there is a *naked* singularity on (at least) one side of the wall. We further generalize the solutions with an arbitrary dilaton coupling function  $f(\phi)$  for the dilaton coupling to the matter potential. For the functions which have an extremum  $df/d\phi|_{\phi_0} = 0$ , *i.e.* self-dual functions, one finds solutions with singularity-free vacuum domain wall space-times. However, it turns out that only non- and ultra-extreme walls of this type are dynamically stable. We also comment on the effects of a dilaton mass, which in basic theory can be induced as a non-perturbative effect. Such a mass (or any other attractive self-interaction) does not alter the space-time sufficiently to remove the naked singularity.

The nature of the space-time for non- and ultra-extreme dilatonic domain walls, which possesses naked singularities, poses serious constraints on the phenomenological viability of theories with dilaton fields, including a large class of  $N = 1$  supergravity theories as well as the effective low energy theory from superstrings.

The paper is organized as follows. In Section II we spell out the formalism for the study of space-time of dilatonic domain walls in the thin wall approximation. Section III contains the formalism for embedding extreme solutions into tree level  $N = 1$  supergravity theory. In Section IV we present the explicit form of the extreme dilatonic domain wall system and comment on their physical properties. Then, in Section V we analyse the non- and ultra-extreme solutions. Our results are summarized and discussed in the concluding

## II. DILATONIC DOMAIN WALLS: THIN WALL FORMALISM

A generic feature of the dilaton is that it couples coherently to matter sources, like kinetic energy of the gauge fields and the potential associated with the scalar matter fields. In the case of a domain wall, due to the dilaton coupling to the potential of the wall-forming scalar field, the dilaton will in general vary with the distance from the wall. This in turn implies that the gravitational field outside the wall is determined not only by the direct gravitational effect of the wall, but also indirectly through the effects of the varying dilaton, which in turn change the nature of space-time on either side of the wall.

Our main goal is to understand the global structure of the induced space-times for such dilatonic domain wall configurations. The ones of most interest are the walls where on one side the dilaton is constant and space-time is Minkowskian. The other side would involve a new space-time with a varying dilaton field. In this paper we shall primarily concentrate on such domain walls, however, we shall also discuss domain walls, which are reflection-symmetric.

In this Section we present the thin wall formalism for the study of the induced space-time outside the wall region for any type (extreme, non- and ultra-extreme) of dilatonic domain walls. The Lagrangian for the domain wall system is specified in Subsection II A. In Subsection II B we specify the Ansätze for the metric and dilaton field and the resulting field equations. The boundary conditions, in the thin wall approximation, are given in Subsection II C. Finally, in Subsection II D we point out a correspondence between the dilatonic domain wall systems and cosmological solutions with a dilaton field.

### A. Lagrangian

The starting point for the study of the dilatonic domain walls is the bosonic part of the action,  $S \equiv \int \mathcal{L} \sqrt{-g} d^4x$ , for the space-time metric  $g_{\mu\nu}$ , the matter field  $\tau$  responsible for the formation of the wall<sup>1</sup> and the dilaton field  $\phi$ , which couples to the matter potential. In the Einstein frame the Lagrangian for the Einstein-dilaton-matter system is:<sup>2</sup>

$$\mathcal{L} = -\frac{1}{2}R + \partial_\mu \tau \partial^\mu \tau + \partial_\mu \phi \partial^\mu \phi - V(\phi, \tau). \quad (1)$$

The potential  $V(\phi, \tau)$  is of the form:

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<sup>1</sup>For the sake of simplicity we introduce only one scalar field responsible for the formation of the wall. We also assume that the wall has no charge, and thus the gauge fields are turned off as well.

<sup>2</sup>Throughout the paper we use units such that  $\kappa \equiv 8\pi G = c = 1$ . Our sign convention for the metric, the Riemann tensor, and the Einstein tensor, is of the type  $-++$  as classified by Misner, Thorne and Wheeler [21].

$$V(\phi, \tau) = f(\phi)V_0(\tau) + \widehat{V}(\phi). \quad (2)$$

The dilaton modulates the matter potential  $V_0(\tau)$  with the function  $f(\phi)$  which we shall denote “the dilaton coupling”. For the sake of generality we have also added a self-interaction term  $\widehat{V}(\phi)$  to the potential (2). This term is not there in the original theory. It is, however, believed to be generated after dynamical (super)symmetry breaking, and it is responsible for giving the dilaton a mass.

In general, the potential (2) need not have a supersymmetric embedding. In this case the walls are not static, in general. Due to the complexity of the field equations, we address the space-time properties in the thin wall approximation, only. Namely, in this case we treat the wall as infinitely thin. It is located at, say,  $z = 0$  (in the rest frame of the wall). Inside the wall, the matter field  $\tau$ , which makes up the wall, yields the stress-energy of the wall with the following  $\delta$ -function contribution:

$$T^\mu_\nu = \delta(z) \sigma \text{diag}(1, 1, 1, 0) \quad (3)$$

where  $\sigma$  is the energy density per unit area of the wall. Outside the wall the matter field  $\tau$  has *no kinetic energy*, and thus its contribution to the energy-momentum tensor on the two sides is given by the constant potentials  $V_0(\tau_1)$  and  $V_0(\tau_2)$ , respectively. Across the thin wall region, both the metric and the dilaton are continuous functions; however, their derivatives are discontinuous. Since the dilaton couples to the matter potential  $V_0$ , on each side of the wall,  $V_0$  becomes a factor in an *effective potential* for the dilaton:

$$V(\phi) = f(\phi)V_0(\tau_i) + \widehat{V}(\phi) \quad (4)$$

where  $\tau_i$  is given by  $\tau_1$  and  $\tau_2$  on the two sides of the wall. The Lagrangian for the dilaton-metric system outside the wall is of the form:

$$\mathcal{L} = -\frac{1}{2}R + \partial_\mu \phi \partial^\mu \phi - V(\phi). \quad (5)$$

In the following Subsection we discuss the field Ansätze, the field equations, and the boundary conditions.

## B. Symmetries and field equations

The domain wall configurations are most conveniently described in the co-moving frame of the wall system. With the requirement of homogeneity, isotropy, boost invariance, and geodesic completeness of the space-time intrinsic to the wall, and the constraint that the same symmetries hold in the hypersurfaces parallel to the wall, the line-element is given as [3]:

$$ds^2 = e^{2a(z)} \left( dt^2 - dz^2 - \beta^{-2} \cosh^2 \beta t \, d\Omega_2^2 \right), \quad (6a)$$

where

$$\beta^{-2} d\Omega_2^2 \equiv \begin{cases} \beta^{-2} d\theta^2 + \beta^{-2} \sin^2 \theta \, d\varphi^2 & \text{if } \beta \neq 0 \\ dx^2 + dy^2 & \text{if } \beta = 0. \end{cases} \quad (6b)$$

The  $z$ -coordinate maps the direction normal to the domain wall. For later convenience the conformal factor in Eq. (6a) is chosen to be of the exponential type  $e^{2a(z)}$ . Note that  $\beta$  parametrizes a deviation of the domain wall configuration from the corresponding planar, extreme wall (with  $\beta = 0$ ). In accordance with the symmetries of the metric, the dilaton field  $\phi$  is a function of  $z$ , only. In other words, the dilaton field is “tied” to the wall system and can thus vary only in a spatial direction perpendicular to the wall.

The Einstein tensor metric Ansatz (6a) is

$$\begin{aligned} G^z_z &= 3 (\beta^2 - a'^2) e^{-2a} \\ G^i_i &= (\beta^2 - a'^2 - 2a'') e^{-2a} \end{aligned} \quad (7)$$

where  $a' \equiv da/dz$  and the index  $i$  stands for a coordinate  $x^i \in \{t, x, y\}$ .

The corresponding energy-momentum tensor on either side of the wall is of the form

$$\begin{aligned} T^z_z &= V(\phi) - \phi'^2 e^{-2a} \\ T^i_i &= V(\phi) + \phi'^2 e^{-2a}, \end{aligned} \quad (8)$$

where  $V(\phi)$  is the effective ‘dilaton potential’ on either side of the wall as defined in Eq. (4) and  $\phi' \equiv d\phi/dz$ . Einstein’s field equations and the second Bianchi identity then lead to the following set of field equations

$$e^{2a} V(\phi) + 2\phi'^2 + 3a'' = 0 \quad (9a)$$

$$-e^{2a} \frac{\partial V(\phi)}{\partial \phi} + 4a'\phi' + 2\phi'' = 0 \quad (9b)$$

$$3\beta^2 - e^{2a} V(\phi) - 3a'^2 + \phi'^2 = 0. \quad (9c)$$

Eq. (9b), which is the energy-momentum conservation law, is identical to the field equation obtained by varying the action with respect to the dilaton. Hence, there are only two independent field equations. Eq. (9c) can be used to determine the boundary conditions.

### C. Boundary conditions

For the solutions on each side of the wall, Israel’s matching conditions [22] relate the energy density (3) of the wall to the discontinuity of the first-order derivatives of the metric in the direction transverse to the singular surface (see Ref. [3] for details<sup>3</sup>):

$$\sigma = 2\zeta_1 a'|_{0-} - 2\zeta_2 a'|_{0+} \quad (10)$$

where  $a' = da/dz$ . Here,  $\zeta$  is a sign factor which is  $+1$  if  $a' > 0$  and  $-1$  if  $a' < 0$  on the corresponding side of the wall. Without loss of generality, we have normalized the metric coefficient at the wall to be  $a(0) = 0$ .

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<sup>3</sup>In Ref. [3] the metric coefficient  $A(z)$  is related to  $a(z)$  by  $A(z) = e^{2a(z)}$ .

Similarly, integrating the equation of motion for the scalar field across the wall region and employing the form (3) of the stress-energy associated with the wall region, one finds that the discontinuity in the first-order derivative of the dilaton at the wall surface is given by:

$$\phi'|_{o^+} - \phi'|_{o^-} = \int_{0^-}^{0^+} e^{2a} V' dz = \frac{\sigma}{2} \frac{\partial \ln[f(\phi)]}{\partial \phi} \Big|_{\phi_0}. \quad (11)$$

Here, again, we have used the fact that the dilaton field is continuous across the wall, with the value  $\phi(0) = \phi_0$ .<sup>4</sup>

In the case when the solution for the metric coefficient and the dilaton is known on one side, one can use Israel's matching condition (10) and the dilaton matching condition (11) to find the boundary condition for the metric and the dilaton on the other side of the wall, and thus the form of the solution on the other side of the wall. In the case of Type I walls, the solution on the side of the wall with  $V_0(\tau_2) = 0$  corresponds to the Minkowski space-time with a *constant* dilaton field. Then, the matching conditions (10) and (11) can be used to determine the solution on the other side of the wall.

In the case of reflection-symmetric walls, the matching conditions (10) and (11) fix the boundary conditions on both sides of the wall. This enables one to find the reflection-symmetric solutions.

As we shall see in the subsequent sections, analytic solutions outside the wall region have been obtained only in the case of extreme (supersymmetric) walls. For the non-extreme walls we have obtained only numerical solutions, in general. Therefore, the above boundary conditions for the Type I and the reflection symmetric walls have been used to determine the boundary conditions for our numerical integrations.

#### D. Relationship between the dilatonic domain wall system and a cosmological model

In this subsection we show that the Einstein-dilaton system outside the wall is equivalent to that of an Einstein-dilaton Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. Formally, one can flip the wall into a space-like hyperspace by a complex coordinate transformation:

$$z \rightarrow \eta, \quad \cosh \beta t \rightarrow i\beta r. \quad (12)$$

If one regards the new coordinates as real, then  $\eta$  becomes the conformal time and  $r$  a spatial coordinate in a metric with opposite signature:  $(-, +, +, +)$ . Changing the sign of the metric implies a change of sign of the curvature scalar,  $R$ , and of all kinetic energy terms in the Lagrangian (5). Because the overall sign of the total Lagrangian is arbitrary, one can change back the sign of the metric, if one at the same time changes the sign of the potential  $V(\phi)$ . Thus, the complex coordinate transformation (12) *maps the domain wall system onto*

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<sup>4</sup>In the thin wall approximation the self-interaction potential  $\hat{V}(\phi)$  of Eq. (2) does not contribute to the energy-momentum (3) of the wall.



a cosmological model having a potential with the opposite sign. As a result, the line-element takes the form

$$ds^2 = e^{2a(\eta)} \left[ d\eta^2 - \frac{dr^2}{1 + \beta^2 r^2} - r^2 d\Omega_2^2 \right]. \quad (13)$$

This is a FLRW line-element where the spatial curvature is  $k = -\beta^2$ .

The equivalence of the Einstein-dilaton system outside the wall with the dilaton-FLRW cosmology (by using the coordinate transformations (12), as well as identifying  $V(\phi) \rightarrow -V_c(\phi)$  and  $\beta^2 \rightarrow -k$ ) proves useful, because it allows us to carry over results from the corresponding cosmological studies. In the cosmological picture the domain wall is a space-like hyper-space. It could be interpreted as representing a phase transition taking place simultaneously throughout the whole universe. In our case, the boundary conditions at this hyper-space are fixed by the boundary conditions of the domain wall.

In addition, we have found it useful to compare the Einstein-dilaton system outside the wall with the evolution of corresponding well-known perfect fluid cosmologies and to compute the corresponding effective equation of state for the dilaton. In terms of a perfect fluid description, the energy-momentum tensor is of the form:

$$\begin{aligned} T^\eta_\eta &= V_c(\phi) + \dot{\phi}^2 e^{-2a} \\ T^i_i &= V_c(\phi) - \dot{\phi}^2 e^{-2a}, \end{aligned} \quad (14)$$

where  $\eta$  is the conformal time and  $\dot{\phi} = d\phi/d\eta$ . Here the index  $i$  refers to the three spatial coordinates. Note that in expression (14) the sign of  $V_c(\phi)$  is reversed with respect to the potential of Eq. (8). The expressions (14) correspond to an energy density

$$\rho \equiv T^\eta_\eta = \dot{\phi}^2 e^{-2a} + V_c(\phi) \quad (15a)$$

and a pressure

$$p \equiv -T^i_i = \dot{\phi}^2 e^{-2a} - V_c(\phi) \quad (15b)$$

of a perfect fluid with a four-velocity  $u^\mu = e^{-a} \delta^\mu_\eta$ .

It is conventional to parametrize the equation of state of a perfect fluid by the  $\gamma$ -parameter:  $p = (\gamma - 1)\rho$ . The following values of  $\gamma$  are singled out:  $\gamma = 0$  corresponds to the equation of state of a cosmological constant;  $\gamma = 2/3$  is the equation of state of a cloud of strings;  $\gamma = 1$  represents dust (non-relativistic cloud of particles);  $\gamma = 4/3$  is radiation (ultra-relativistic matter); and  $\gamma = 2$  corresponds to a Zel'dovich fluid (maximally stiff matter). All physical equations of state are confined to the range  $0 \leq \gamma \leq 2$ . This is also the range covered by a minimally coupled scalar field  $\phi$ :

$$\gamma = \frac{2\dot{\phi}^2 e^{-2a}}{\dot{\phi}^2 e^{-2a} + V_c(\phi)}. \quad (16)$$

It has  $\gamma = 2$  if the kinetic energy dominates, and  $\gamma = 0$  if the potential energy dominates. Matter satisfying an equation of state with  $\gamma < 1$  has negative pressure. If  $\gamma < 2/3$ , then the repulsive gravitational effect of the negative pressure is greater than the attractive gravitational effect of the energy density. Matter obeying such an equation of state is therefore a source of repulsive gravity. An effective equation of state of this kind is a necessary ingredient in inflationary universe models.

### III. SUPERSYMMETRIC EMBEDDING

Solutions of the theory specified by the Lagrangian (1) can be embedded in a supergravity theory. In this section we spell out the formalism for an embedding of dilatonic domain walls into the corresponding tree level  $N = 1$  supergravity theory. Extreme domain walls turn out to be static, planar configurations interpolating between *supersymmetric* minima of the corresponding supergravity potential. Such configurations satisfy the corresponding Bogomol'nyi equations for *any thickness* of the wall and the energy density of the wall, which can be precisely defined only in the thin wall approximation, saturates the corresponding Bogomol'nyi bound. In the thin wall approximation the Einstein-dilaton system outside the wall is described by the formalism spelled out in the previous section with the non-extremality parameter  $\beta = 0$ .

The results presented in this section is a generalization of the previous work on supergravity walls [4] without the dilaton, and dilatonic supergravity walls [19,20] with a special form of the function  $f(\phi) = e^{2\sqrt{\alpha}\phi}$  in the dilaton effective potential (4). The latter ones arise in  $N = 1$  supergravity with a general coupling of the linear supermultiplet.

In Subsection III A the supersymmetric embedding of extreme dilatonic domain walls is realized within  $N = 1$  supergravity theory with (gauge neutral) chiral superfields whose Kähler and super-potential are constrained. Namely, the Lagrangian in such a  $N = 1$  supergravity contains (gauge neutral) matter chiral-superfield  $\mathcal{T}$ , whose scalar component  $T$  is responsible for a formation of the wall. In addition, there is a chiral superfield  $\mathcal{S}$ , which has no superpotential and whose Kähler potential decouples from the one of  $\mathcal{T}$ . In turn, the scalar component  $S$  of the chiral superfield  $\mathcal{S}$  acts as the dilaton field, which couples to the matter potential. We derive the Bogomol'nyi bound and Bogomol'nyi equations (Killing spinor equations) for the extreme dilatonic wall solutions in Subsections III B and III C, respectively. The latter subsection also contains a classification of the extreme dilatonic domain walls.

#### A. $N = 1$ Supergravity Lagrangian

In supergravity theories the bosonic Lagrangian (1) arises in  $N = 1$  supergravity when the gauge-singlet matter chiral superfield  $\mathcal{T}$  has a non-zero Kähler potential (real function of chiral superfields)  $K_{\text{matt}}(\mathcal{T}, \mathcal{T}^*)$  as well as a non-zero superpotential (holomorphic function of the fields)  $W_{\text{matt}}(\mathcal{T})$ .<sup>5</sup> In addition, there is a chiral superfield  $\mathcal{S}$  with the Kähler potential  $K_{\text{dil}}(\mathcal{S}, \mathcal{S}^*)$ , which *does not* couple to the matter Kähler potential, and with *no* superpotential ( $W_{\text{dil}}(\mathcal{S}) = 0$ ). The bosonic part of the Lagrangian without the gauge fields is then fully specified by:

$$\begin{aligned} K &= K_{\text{dil}}(\mathcal{S}, \mathcal{S}^*) + K_{\text{matt}}(\mathcal{T}, \mathcal{T}^*), \\ W &= W_{\text{matt}}(\mathcal{T}), \end{aligned} \tag{17}$$

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<sup>5</sup>For the sake of simplicity we assume that the wall is formed by a scalar component of one chiral superfield only.

and it is given by

$$\mathcal{L} = -\frac{1}{2}R + K_{TT^*}\partial_\mu T \partial^\mu T^* + K_{SS^*}\partial_\mu S \partial^\mu S^* - V \quad (18)$$

where the potential is

$$V = e^K \left[ |D_T W|^2 K^{TT^*} - (3 - |K_S|^2 K^{SS^*}) |W|^2 \right]. \quad (19)$$

Here,  $T$  and  $S$  are the scalar components of the chiral superfields  $\mathcal{T}$  and  $\mathcal{S}$ , respectively;  $K_T = \partial_T K$  and  $K_{TT^*} = \partial_T \partial_{T^*} K$  is the positive definite Kähler metric, and  $D_T W = \partial_T W + K_T W$ .

We assume that the matter part of the potential (19) has isolated minima and the matter field  $T$  is responsible for the formation of the wall. When the potential (19) has isolated supersymmetric minima, *i.e.* when  $D_T W|_{T_{1,2}} = 0$ , there are extreme walls, which are *static* configurations interpolating between these minima.

Because of the restricted form of the Kähler potential and the superpotential (17), the form of the potential (19) resembles closely, although not completely, the form of the potential in Eq. (2) with  $\widehat{V}(\phi) = 0$ . In other words, a supersymmetric embedding of dilatonic walls requires a specific type of potential (19). Inside the wall region, such a potential is in general not of the simple form (2). Outside the wall region, however, Eq. (19) *does* reduce to the effective dilaton potential of the type (4). In particular, in the case of the isolated supersymmetric minima, *i.e.*  $D_T W|_{T_{1,2}} = 0$ , the potential (19) outside the wall *is* of the form of the effective dilaton potential (4) with

$$\begin{aligned} f(\phi) &= e^{K_{\text{dil}}(S, S^*)} (|K_S|^2 K^{SS^*} - 3), \quad \widehat{V} = 0, \\ V_0(\tau_{1,2}) &= \left( e^{K_{\text{mat}}} |W|^2 \right)_{T_{1,2}}. \end{aligned} \quad (20)$$

A particularly interesting case is  $N = 1$  supergravity theory with a matter chiral superfield and a linear supermultiplet [23]. In the Kähler superspace formalism the dilaton linear supermultiplet can be expressed in terms of a chiral supermultiplet  $\mathcal{S}$  with the Kähler potential [23]:

$$K_{\text{dil}}(S, S^*) = -\alpha \ln(S + S^*). \quad (21)$$

With  $S$ , the scalar component of  $\mathcal{S}$ , written as  $S = e^{-2\phi/\sqrt{\alpha}} + i\mathcal{A}$ , the potential in Eq. (2) is related to that of Eq. (19) through:

$$\begin{aligned} f(\phi) &= e^{2\sqrt{\alpha}\phi}, \quad \widehat{V} = 0, \\ V_0 &= e^{K_M} \left[ |D_T W|^2 K^{TT^*} - (3 - \alpha) |W|^2 \right]. \end{aligned} \quad (22)$$

On the other hand, one is also interested in the self-dual Kähler potential where  $K_{\text{dil}}$  has an extremum for finite  $S$ . Such a Kähler potential can be motivated by assuming strong-weak (dilaton) coupling invariance of the theory.

In the following subsections we shall derive the Bogomol'nyi bound on the energy density and the Killing spinor equations for supersymmetric (extreme) domain wall configurations.

## B. Bogomol'nyi bound

In order to derive the corresponding Killing spinor equations and the Bogomol'nyi bounds for dilatonic domain walls, we use the technique of the generalized Israel-Nester-Witten form [26], which was originally applied to the study of ordinary supergravity walls [4]. The results here are somewhat more general than those for extreme walls with an exponential dilaton coupling [19,20].<sup>6</sup> In addition, a generalization of the results to more than one dilaton field is straightforward, as long as the dilatons have no superpotential and a Kähler potential decoupled from the one of the matter field(s). Since the extreme domain walls are planar and infinite, we shall derive the Bogomol'nyi bound for the energy per unit area of the wall. Note also that a precise definition of the energy density of the wall is possible only in the thin wall approximation, namely, when the “interior” and the “exterior” regions of the wall are clearly separated.

We consider a generalized Nester form [26]:

$$N^{\mu\nu} = \bar{\epsilon} \gamma^{\mu\nu\rho} \widehat{\nabla}_\rho \epsilon \quad (23)$$

where  $\epsilon$  is a Majorana spinor. Here  $\widehat{\nabla}_\rho \epsilon \equiv \delta_\epsilon \psi_\rho$  and  $\widehat{\nabla}_\rho = 2\nabla_\rho + Q_\rho$ , where  $Q_\rho = ie^{\frac{K}{2}} (\Re(W) + \gamma^5 \Im(W)) \gamma_\rho - \gamma^5 \Im(K_T \partial_\rho T) - \gamma^5 \Im(K_S \partial_\rho S)$  and  $\nabla_\mu \epsilon = (\partial_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab}) \epsilon$ ;  $\psi_\rho$  is the spin 3/2 gravitino field. Therefore, the explicit expression for Nester's form is:<sup>7</sup>

$$N^{\mu\nu} = \bar{\epsilon} \gamma^{\mu\nu\rho} \left[ 2\nabla_\rho + ie^{\frac{K}{2}} (\Re(W) + \gamma^5 \Im(W)) \gamma_\rho - \Im(K_T \partial_\rho T) \gamma^5 - \Im(K_S \partial_\rho S) \gamma^5 \right] \epsilon. \quad (24)$$

Stokes' theorem ensures the following relationship:

$$\int_{\partial\Sigma} N^{\mu\nu} d\Sigma_{\mu\nu} = 2 \int_\Sigma \nabla_\nu N^{\mu\nu} d\Sigma_\mu \quad (25)$$

where  $\Sigma$  is a space-like hypersurface.

After a lengthy calculation (for details related to the derivation of the expression below, see Appendices in Ref. [4]), the volume integral yields in our case:

$$2 \int_\Sigma \nabla_\nu N^{\mu\nu} d\Sigma_\mu = \int \left[ \overline{\widehat{\nabla}_\nu \epsilon} \gamma^{\mu\nu\rho} \hat{\nabla}_\rho \epsilon + K_{TT^*} \overline{\delta_\epsilon \chi} \gamma^\mu \delta_\epsilon \chi + K_{SS^*} \overline{\delta_\epsilon \eta} \gamma^\mu \delta_\epsilon \eta + (G^{\mu\nu} - T^{\mu\nu}) \bar{\epsilon} \gamma_\nu \epsilon \right] d\Sigma_\mu \geq 0, \quad (26)$$

where  $\delta_\epsilon \chi$  and  $\delta_\epsilon \eta$  are the supersymmetry transformations of fermionic partners  $\chi$  and  $\eta$  to the matter field  $T$  and the dilaton field  $S$ , respectively;  $T^{\mu\nu}$  is the energy-momentum tensor

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<sup>6</sup>Analogous procedures were followed in the derivation of the Bogomol'nyi bounds for the mass of the corresponding charged black holes [24,25].

<sup>7</sup>We use the conventions:  $\gamma^\mu = e^\mu_a \gamma^a$  where  $\gamma^a$  are the flat space-time Dirac matrices satisfying  $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ ,  $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ ;  $e^a_\mu e^\mu_b = \delta^a_b$ ;  $a = 0, \dots, 3$ ;  $\mu = t, x, y, z$ .

and  $G^{\mu\nu}$  is the Einstein tensor. The first term in Eq. (26) is non-negative, provided the spinor  $\epsilon$  satisfies the (modified) Witten condition, *i.e.*  $\mathbf{n} \cdot \widehat{\nabla} \epsilon = 0$  ( $\mathbf{n}$  is the four-vector normal to  $\Sigma$ ). The Kähler metric coefficients  $K_{TT^*}$  and  $K_{SS^*}$  are positive definite, and thus the second and the third terms in Eq. (26) are non-negative as well. The last term in Eq. (26) is zero due to Einstein's equations. Thus, the integrand in Eq. (26) is always non-negative and it is zero if and only if the supersymmetry transformations on the gravitino  $\psi_\rho$  as well as on  $\chi$  and  $\eta$  vanish, *i.e.* if the configurations are supersymmetric.

The surface integral of Nester's form in Eq. (25) yields the corresponding Bogomol'nyi bound for the energy associated with the configuration. We shall derive the energy per unit area (energy density) of the wall by evaluating the corresponding density of the surface integral in Eq. (25). Such a bound can be derived precisely only in the thin wall approximation, because the region inside the wall must be clearly separated from the region outside the wall in order for its energy density to be well defined.

Within the above assumptions the space-like hyper-surface  $\Sigma$  extends in the  $z$ -direction only. The measure is  $d\Sigma_\mu = (d\Sigma_t, 0, 0, 0)$  and  $d\Sigma_t = \sqrt{|g_{tt}g_{zz}|}dz$ . For a thin wall located at  $z = 0$  the boundary associated with the surface integral are then the two points at  $0^+$  and  $0^-$ . In addition, at the location of the thin wall, the metric coefficient  $a(0) = 0$ , and the dilaton has the value  $S(0) = S_0$ . Thus, the corresponding density of the surface integral of Nester's form in Eq. (25) is of the form:

$$\bar{\epsilon}_0 \gamma^0 \epsilon_0 \sigma + \bar{\epsilon}_0 \gamma^{03} e^{K/2} [\Re(W) + \gamma^5 \Im(W)] \epsilon_0 \Big|_{0^-}^{0^+}. \quad (27)$$

The spinor  $\epsilon_0$  is defined at the boundaries  $z = 0^+$  and  $z = 0^-$  of the wall. In the first term we have used the fact that for the thin wall the magnitude of the spinor components does not change. The first term of the surface integral (27) of the Nester's form (23) can then be identified with the energy density of the wall. The second term corresponds to the topological charge density  $C$  evaluated on both sides of the wall. Positivity of the volume integral (26) translates through Eq. (25) into the corresponding Bogomol'nyi bound for the energy density of a thin wall:

$$\sigma \geq |C|, \quad (28)$$

which is saturated if and only if the bosonic background is supersymmetric.

In the following subsection we shall evaluate the explicit phase factors by which the components of the  $\epsilon_0$  spinor change at the wall boundaries for the case of extreme solutions. These phase factors will in turn allow us to obtain the explicit form of  $\sigma_{\text{ext}} = |C|$ .

### C. Killing spinor equations

We now write down explicit Killing spinor equations, *i.e.*  $\delta\psi_\mu = \delta\chi = \delta\eta = 0$ . Killing spinor equations are satisfied by supersymmetric, static configurations. With the metric Ansatz (6) with  $\beta = 0$ :

$$ds^2 = e^{2a(z)} (dt^2 - dz^2 - dx^2 - dy^2) \quad (29)$$

and  $T(z)$  and  $S(z)$  being functions only of  $z$ , the Killing spinor equations are of the form:

$$\delta\chi = -\sqrt{2} \left[ e^{\frac{K}{2}} K^{TT^*} \left( \Re(D_T W) + \gamma^5 \Im(D_T W) \right) + i e^a \left( \Re(\partial_z T) + \gamma^5 \Im(\partial_z T) \right) \gamma^3 \right] \epsilon \quad (30a)$$

$$\delta\eta = -\sqrt{2} \left[ e^{\frac{K}{2}} K^{SS^*} \left( \Re(K_S W) + \gamma^5 \Im(K_S W) \right) + i e^a \left( \Re(\partial_z S) + \gamma^5 \Im(\partial_z S) \right) \gamma^3 \right] \epsilon \quad (30b)$$

$$\delta\psi_x = \left[ -\gamma^1 \gamma^3 \partial_z a - i \gamma^1 e^{(a+\frac{K}{2})} \left( \Re W + \gamma^5 \Im W \right) \right] \epsilon \quad (30c)$$

$$\delta\psi_y = \left[ -\gamma^2 \gamma^3 \partial_z a - i \gamma^2 e^{(a+\frac{K}{2})} \left( \Re W + \gamma^5 \Im W \right) \right] \epsilon \quad (30d)$$

$$\delta\psi_z = \left[ 2\partial_z - i \gamma^3 e^{(a+\frac{K}{2})} \left( \Re W + \gamma^5 \Im W \right) - \gamma^5 \Im(K_T \partial_z T + K_S \partial_z S) \right] \epsilon \quad (30e)$$

$$\delta\psi_t = \left[ \gamma^0 \gamma^3 \partial_z a + i \gamma^0 e^{(a+\frac{K}{2})} \left( \Re W + \gamma^5 \Im W \right) \right] \epsilon. \quad (30f)$$

We have assumed that the Majorana spinor  $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_2^*, -\epsilon_1^*)$  does not depend on  $x^i \in \{t, x, y\}$ . Note that in Eqs. (30) the Kähler potential  $K = K_{\text{dil}}(S, S^*) + K_{\text{matt}}(T, T^*)$  is separable and  $W = W(T)$ , cf. Eq. (18).

The vanishing of the above expressions yields first-order differential equations<sup>8</sup> (self-dual or Bogomol'nyi equations) for the metric coefficient  $a(z)$ ,  $T(z)$  and  $S(z)$  as well as the constraint on the spinor  $\epsilon$ . The field equations are of the form:

$$0 = \Im \left( \partial_z T \frac{D_T W}{W} \right) \quad (31a)$$

$$\partial_z T = \zeta e^{(a+\frac{K}{2})} |W| K^{TT^*} \frac{D_T W}{W^*} \quad (31b)$$

$$\partial_z a = \zeta e^{(a+\frac{K}{2})} |W| \quad (31c)$$

$$\partial_z S = -\zeta e^{(a+\frac{K}{2})} |W| K^{SS^*} K_{S^*}. \quad (31d)$$

Here  $\zeta = \pm 1$  and it can change sign only when  $W$  crosses zero. There is another constraint on the “field geodesic” motion of the dilaton field, namely  $\Im(K_S \partial_z S) = 0$ . However, by multiplying Eq. (31d) by  $K_S$ , this constraint is seen to be automatically satisfied. In this case the right-hand side of the equation is real, since  $K^{SS^*} > 0$  is real and  $K_{S^*} = (K_S)^*$ .

Eqs. (31a) and (31b) describe the evolution of the matter field  $T = T(z)$  with  $z$ . By now, Eq. (31a) is a familiar “field geodesic” equation, which determines the path of the complex scalar field  $T$  in the complex plane between the two minima  $T_1$  and  $T_2$  of the matter potential. It can always be satisfied for Type I walls (those with  $W(T_1) = 0$ ). Eq. (31b) governs the change of the  $T$  field with coordinate  $z$  along this path (analogous to the field  $\tau$  in Section II).

Eqs. (31c) and (31d) determine the evolution of the metric coefficient  $a(z)$  and the complex field  $S$ . These two equations imply another interesting relation between the dilaton Kähler potential  $K_{\text{dil}}(S, S^*)$  and  $a(z)$ :

$$2K^{SS^*} |K_S|^2 \partial_z a + \partial_z K_{\text{dil}} = 0. \quad (32)$$

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<sup>8</sup>Eqs. (30) equal zero can be viewed as “square roots” of the corresponding Einstein and Euler-Lagrange equations; they provide a particular solution of the equations of motion which saturate the Bogomol'nyi bound (28). The existence of such static wall solutions is due to the constrained form of the matter potential in  $N = 1$  supergravity theory. Note also that in the thin wall approximation one can explicitly solve the Einstein equations for  $a(z)$  and the Euler-Lagrange equation for  $S(z)$  outside the wall and then match the solution for  $a(z)$  and  $S(z)$  across the wall region.

In addition, the Killing spinor equations (30) impose a constraint on the phase of the Majorana spinor. Namely, the solution for the Killing spinor component is of the form:

$$\epsilon_1 = e^{i\theta} \epsilon_2^* = \mathcal{C} e^{\frac{(a+i\theta)}{2}}, \quad (33)$$

where the phase  $\theta(z)$  satisfies the following equation:

$$\partial_z \theta = -\Im(K_T \partial_z T). \quad (34)$$

The constant  $\mathcal{C}$  can be set to 1/2 for the Majorana spinors normalized as  $\epsilon^\dagger \epsilon = 1$ . The constraint (33) on the Killing spinor  $\epsilon$  in turn implies that the extreme configurations preserve “ $N = \frac{1}{2}$ ” of the original  $N = 1$  supersymmetry.

The energy density of the wall is determined by setting Eq. (27) to zero. With the explicit form for the Killing spinor components (33), Eq. (27) yields:

$$\begin{aligned} \sigma_{\text{ext}} = |C| &= 2 \left| \left( \zeta e^{\frac{K}{2}} W \right)_{z=0^+} - \left( \zeta e^{\frac{K}{2}} W \right)_{z=0^-} \right| \\ &= 2e^{\frac{K_{\text{dil}}(S_0, S_0^*)}{2}} \left| \left( \zeta e^{\frac{K_{\text{matt}}}{2}} W \right)_{z=0^+} - \left( \zeta e^{\frac{K_{\text{matt}}}{2}} W \right)_{z=0^-} \right|. \end{aligned} \quad (35)$$

Here the subscript  $z = 0^\pm$  refers to either side of the wall. Without loss of generality we have normalized  $a(0) = 0$  and set  $S(0) = S_0$ .

*Classification of extreme domain wall solutions:* Solutions to the Bogomol’nyi equations (31) fall into three types, depending on whether  $W(T)$  crosses zero or not along the wall trajectory:

**Type I** walls correspond to those where on one side of the wall, say for  $z > 0$ ,  $W(T_1) = 0$ . In this case the energy density of the wall is of the form:  $\sigma_{\text{ext}} = 2 \left| e^{\frac{K}{2}} W \right|_{z=0^-}$ . Note that in this case the side of the wall with  $z > 0$  corresponds to the Minkowski space-time with a constant  $S$ .

**Type II** walls correspond to the walls with  $W(T)$  crossing zero somewhere along the wall trajectory. In this case  $\zeta$  changes sign at  $W = 0$ . The energy density of the wall is specified by:  $\sigma_{\text{ext}} = 2 \left| e^{\frac{K}{2}} W \right|_{z=0^+} + \left| e^{\frac{K}{2}} W \right|_{z=0^-}$ . Reflection symmetric walls fall into this class.

**Type III** walls correspond to the walls where  $W(T) \neq 0$  everywhere in the domain wall background. In this  $\zeta$  does not change sign. The energy density of such walls is:  $\sigma_{\text{ext}} = 2 \left| \left| e^{\frac{K}{2}} W \right|_{z=0^+} - \left| e^{\frac{K}{2}} W \right|_{z=0^-} \right|$ .

In the following section we shall concentrate on the explicit form of the extreme solutions using some of the formalism spelled out in the previous two sections.

#### IV. EXPLICIT FORM OF EXTREME SOLUTIONS

This section concentrates on the Einstein-dilaton system outside the extreme domain wall region. The explicit form of the extreme solutions in the thin wall approximation<sup>9</sup> are

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<sup>9</sup>Explicit numerical solutions of Eqs. (31) for a wall of any thickness have the same qualitative features outside the wall region.

presented.

We shall first recapitulate results for extreme walls with  $K_{\text{dil}} = -\alpha \ln(S + S^*)$  [20]. Then we shall study extreme domain walls with self-dual  $K_{\text{dil}}$ , *i.e.*  $K_S|_{S'} = 0$  for some  $S'$ . An example of the latter class corresponds to a solution of a theory with a strong-weak (dilaton) coupling symmetry, *i.e.*,  $SL(2, Z)$  invariance of the dilaton coupling.

Subsection IV A presents the extreme solutions in theories with exponential dilaton coupling. Their physical properties such as the Hawking temperature associated with the horizons and the gravitational mass of the singularities are also discussed. In Subsection IV B the correspondence with cosmological models is used to find the necessary and sufficient condition for the existence of horizons. Subsection IV C comments on the self-dual extreme solutions.

### A. Extreme solutions for exponential dilaton coupling

Let us first consider the gravitational properties of extreme supersymmetric domain walls with  $K_{\text{dil}} = -\alpha \ln(S + S^*)$ , which has been worked out in Refs. [19,20]. The case with  $\alpha = 2$  had earlier been found in the guise of a static plane-symmetric space-time with a conformally coupled scalar field [30,31]. In Ref. [32] this space-time was shown to be induced by a domain wall. Here we shall recapitulate the results and focus on the relationship with cosmological solutions.

With the notation  $S = e^{-2\phi/\sqrt{\alpha}} + i\mathcal{A}'$ , the Bogomol'nyi equations of motion are of the type:

$$0 = \Im \left( \partial_z T \frac{D_T W}{W} \right) \quad (36a)$$

$$\partial_z T = -\zeta 2^{-\frac{\alpha}{2}} e^{(a+\sqrt{\alpha}\phi)} e^{\frac{K_{\text{matt}}}{2}} |W| K^{TT*} \frac{D_T^* W^*}{W^*} \quad (36b)$$

$$\partial_z a = \zeta 2^{-\frac{\alpha}{2}} e^{(a+\sqrt{\alpha}\phi)} e^{\frac{K_{\text{matt}}}{2}} |W| \quad (36c)$$

$$\partial_z \phi = -\zeta \sqrt{\alpha} 2^{-\frac{\alpha}{2}} e^{(a+\sqrt{\alpha}\phi)} e^{\frac{K_{\text{matt}}}{2}} |W|, \quad (36d)$$

and the axion  $\mathcal{A}'$ , the imaginary part of  $S$ , is constant. The energy density of the wall is of the form:

$$\sigma_{\text{ext}} = 2^{1-\frac{\alpha}{2}} e^{\sqrt{\alpha}\phi_0} \left| \left( e^{\frac{K_{\text{matt}}}{2}} W \right)_{z=0^+} \pm \left( e^{\frac{K_{\text{matt}}}{2}} W \right)_{z=0^-} \right| \equiv 2(\chi_1 \pm \chi_2) \quad (37)$$

where we have chosen the boundary condition for  $a(0) = 0$  and  $\phi(0) = \phi_0$ . The  $+$  and  $-$  signs correspond to the Type II and Type III walls, respectively. It is understood that the coordinates are chosen so that  $\chi_2 \leq \chi_1$ . With the choice  $\chi_{1,2} = 2|e^{K/2}W|_{z=0^\mp}$ , the solution of Eqs. (31) and (36) are satisfied with the following choice for  $\zeta$ : on the  $z < 0$  side,  $\zeta = 1$ , while on the  $z > 0$ ,  $\zeta = -1$  for Type II walls and  $\zeta = 1$  for Type III walls.

The solution of the Bogomol'nyi equations (36) outside the wall region, *i.e.* when  $\partial_z T \sim 0$ , are the same as those of the second-order equations (9) with  $\beta = 0$  and the effective dilaton potential of the type of Eq. (4), where

$$f(\phi) = e^{2\sqrt{\alpha}\phi}, \quad V_0(\tau_{1,2}) = -(3 - \alpha)2^{-\alpha} \left( e^{K_{\text{matt}}} |W|^2 \right) \Big|_{T_{1,2}}, \quad \hat{V}(\phi) = 0 \quad (38)$$



on either side of the wall. The value of parameters  $\chi_{1,2}$  in Eq. (37) on either side of the wall is related to  $V_0(\tau_{1,2})$  in the following way

$$\chi_{1,2} \equiv 2^{-\frac{\alpha}{2}} e^{\sqrt{\alpha}\phi_0} \left( e^{\frac{K_{\text{matt}}}{2}} |W| \right) |_{T_{1,2}} = e^{\sqrt{\alpha}\phi_0} \sqrt{-V_0(\tau_{1,2})/(3-\alpha)}. \quad (39)$$

Note that  $\alpha = 3$  corresponds to the point where  $V_0$  changes sign.

The explicit solution on either side of the wall is of the form

$$a = \begin{cases} \chi_1 z, & z < 0 \\ \mp \chi_2 z, & z > 0 \end{cases} \quad (40a)$$

if  $\alpha = 1$  and

$$a = \begin{cases} (\alpha - 1)^{-1} \ln[1 + \chi_1(\alpha - 1)z], & z < 0 \\ (\alpha - 1)^{-1} \ln[1 \mp \chi_2(\alpha - 1)z], & z > 0 \end{cases} \quad (40b)$$

if  $\alpha \neq 1$ . The upper and lower signs of the solutions (40) correspond to the Type II and Type III solutions, respectively. Type I corresponds to the special case with  $\chi_2 = 0$ , *i.e.* those are solutions with Minkowski space-time ( $a = 0$ ) and a constant dilaton on the  $z > 0$  side of the wall.

Note that Eqs. (36c) and (36d) imply that

$$\phi = -\sqrt{\alpha}a \quad (41)$$

everywhere in the domain wall background. Consequently, these solutions are represented by straight lines in the  $(a', \phi')$  phase diagram.

For  $\alpha > 1$  the domain walls have a naked (planar) singularity at  $z = -1/[\chi_1(\alpha - 1)]$  and for Type II walls at  $z = 1/[\chi_2(\alpha - 1)]$  as well. For  $\alpha \leq 1$  the singularity becomes null, *i.e.* it occurs at  $z = -\infty$  and for Type II walls at  $z = \infty$  as well. Note that for  $\alpha < 1$  Type III walls have a coordinate singularity at  $z = 1/[\chi_2(1 - \alpha)]$ . Thus, extreme walls with the “stringy” coupling  $\alpha = 1$  act as a window between the extreme dilatonic walls with naked singularities and those with singularities covered by a horizon.

The evolution of  $a$  for different values of  $\alpha$  is plotted in Fig. 2.

*Temperature and gravitational mass per area:* Static domain wall configurations with space-time singularities are only possible if there is an exact cancellation of the contributions to the gravitational mass coming from the wall, the dilaton field, and from the singularity. Let us consider the case of an extreme Type I wall, *i.e.* a static dilatonic wall with a non-zero vacuum energy on one side (say,  $z < 0$  and  $\chi_1 \neq 0$ ) and a Minkowski space on the other ( $z > 0$  and  $\chi_2 = 0$ ).

We shall employ the concept of gravitational mass per area,  $\Sigma$ , as derived in Ref. [3]. It can be written in the form:

$$\Sigma(z) = \frac{\int_{-\infty}^z \sqrt{-g^{(4)}} dz' (T_t^t - T_z^z - T_x^x - T_y^y) \int dx dy}{\sqrt{g^{(2)}} \int dx dy}. \quad (42)$$

This is a plane-symmetric version of Tolman’s [34] mass formula. The contribution from all sources outside the horizon (or the naked singularity) can be expressed as

$$\Sigma(\infty) = 2a'|_{\infty} - 2a'|_{\text{horizon}}. \quad (43)$$

On the Minkowski side  $a' \equiv da/dz = 0$ , and so  $a'|_{\infty} = 0$ . Hence,  $\Sigma$  is determined by the value of  $a'$  at the horizon. For  $\alpha < 1$ , the gravitational mass per area outside the horizon vanishes. This means that there is no mass beyond the horizon. The Hawking temperature associated with this horizon also vanishes.

For  $\alpha = 1$ , the gravitational mass per area from sources outside the horizon is negative:  $\Sigma(\infty) = -2\chi_1$ . In order to have a vanishing total mass, the mass of the singularity must be  $\Sigma_{\text{singularity}} = 2\chi_1$ , but a proper mathematical description would require a distribution-valued metric. We shall not pursue this issue further here, but we note that a source at the singularity of the Schwarzschild metric has recently been identified in terms of a distributional energy-momentum tensor [35]. The Hawking temperature for the extreme Type I wall with  $\alpha = 1$  is finite:  $T = \chi_1/2\pi$  [20].

For  $\alpha > 1$ , the mass per area outside the singularity is negative and infinite. Accordingly, there must be an infinite positive mass in the singularity, which is then a singularity even in a distributional sense. This naked singularity also has an infinite Hawking temperature.

## B. Correspondence with cosmological solutions

The equivalence of the domain wall solutions on either side of the wall and a class of cosmological solutions implies that we are able to relate the above solutions to known solutions of inflationary cosmology with exponential potentials [27], which were later generalized to higher-dimensional FLRW cosmologies [28]. Properties of general (extreme and non-extreme) scalar field cosmological models and their corresponding phase diagrams were studied in Ref. [29].<sup>10</sup>

Note that after the substitution  $z \rightarrow \eta$  and  $V_0 \rightarrow -V_{0c}$  the Type II solutions and Type III solutions (on the  $z > 0$  side) correspond to contracting and expanding cosmological solutions, respectively. For cosmological models  $\chi_{1,2} \equiv \sqrt{V_{0c}/(3-\alpha)}$ . The value  $\alpha = 3$  corresponds to the point where  $V_{c0}$  changes sign from positive (for  $\alpha < 3$ ) to negative (for  $\alpha > 3$ ).

Since the extreme solutions are characterized by  $\phi = -\sqrt{\alpha}a$  they are represented by straight lines in the  $(\dot{a}, \dot{\phi})$  phase diagram [29].

*Cosmological horizons and domain wall event horizons:* We would now like to relate the nature of the cosmological horizons to the event horizons in the domain wall background. If we write the cosmological line element in the standard form

$$ds^2 = d\tau^2 - R^2(\tau) \left( \frac{dr^2}{1 + \beta^2 r^2} + r^2 d\Omega_2^2 \right), \quad (44)$$

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<sup>10</sup>Note that in the cosmological picture the extreme Type I vacuum domain wall becomes a flat inflationary universe where the inflaton (the wall-forming scalar field in the original picture) rolls down the inflation potential with just the right speed so that it stops at a local maximum corresponding to a vanishing cosmological constant. At this point the universe also stops expanding.

then the *convergence* of the integral

$$I = \int_{\tau_0}^{\tau_1} \frac{d\tau}{R(\tau)} = \int_{\eta_0}^{\eta_1} d\eta \quad (45)$$

in the limit  $\tau_1 \rightarrow \tau_{\max}$  is a necessary and sufficient condition for the existence of a cosmological event horizon [33]. Note, however, that the complex rotation to the domain wall space-time interchanges a space dimension with the time dimension. Because of this, the sufficient and necessary condition for having an event horizon in the domain wall space-time is that

$$I = \int_{\eta_0}^{\eta_{\max}} d\eta = \eta_{\max} - \eta_0 \quad (46)$$

*diverges*. In other words, if there is a singularity at finite  $\eta$ , then this singularity is naked.

*Equation of state:* For  $0 \leq \alpha \leq 3$ , the equation of state is given by

$$\gamma = \frac{2\alpha}{3}. \quad (47)$$

The “stringy” value,  $\alpha = 1$ , is therefore the dividing line between the solutions corresponding to attractive and repulsive equations of state in the cosmological picture. In the domain wall system, this is the dividing line of domain walls with naked singularities ( $\alpha > 1$ ), and domain walls with the singularity hidden behind a horizon ( $\alpha < 1$ ).

For  $\alpha > 3$  the equation of state is time-dependent (see Fig. 1). It approaches  $\gamma = 2$  near the singularity.

### C. Self-dual dilaton coupling

We would also like to address extreme domain wall solutions corresponding to the self-dual dilaton Kähler potential  $K_{\text{dil}}$ . Namely,  $K_{\text{dil}}$  has an extremum ( $K_S|_{S'} \equiv \partial K_{\text{dil}}/\partial S|_{S'} = 0$ ) for some finite  $S = S'$ . The hope is that outside the wall region the dilaton would reach the point  $S = S'$ , and from then on it would remain constant. Such solutions would in turn reduce to singularity-free space-times of vacuum domain walls. We shall see that such a class of extreme solutions is not dynamically stable within  $N = 1$  supergravity theory.

In particular, we shall show that if at  $z \sim 0$ ,  $S(0) = S' + \Delta(0)$ , with  $\Delta(0)$  being an infinitesimal perturbation from the self-dual point,  $S = S'$ , then  $\Delta(z)$  grows indefinitely as  $z \rightarrow -\infty$  and thus the solution with a constant dilaton outside the wall region is not dynamically stable. On the  $z < 0$  side of the wall,  $\Delta(z)$  (and likewise for  $\Delta^*$ ) is a solution of the following equation:

$$\Delta'' - (a' + \zeta \chi_1 e^a) \Delta' + \chi_1^2 (1 - |K_{SS} K^{SS*}|^2) e^{2a} \Delta = 0, \quad (48)$$

where  $\chi_1 = (e^{K/2} W)|_{S', T_1}$  and  $\zeta = 1$  on the  $z < 0$  side of the wall (see conventions spelled out after Eq. (37)). Eq. (48) is obtained by expanding  $S(z) = S' + \Delta(z)$  in Eq. (31d). In the above expansion one has used  $K_S|_{S'} = 0$  and  $K^{SS*} > 0$ .

In Eq. (48) it suffices to use the metric coefficient  $a(z)$ , which is determined as a zeroth-order solution of Eq. (31c). Namely, in Eq. (31) one sets  $S = S'$ , and thus  $a(z) = -\ln(1 -$

$\chi_1 z$ ) is the solution for the anti-de Sitter space-time with the cosmological constant  $\Lambda = -3\chi_1^2$  (see Eq. (40b) with  $\alpha = 0$ ). Eq. (48) for  $\Delta$  can then be rewritten as:

$$\frac{d^2 \Delta}{dy^2} - 2 \frac{d\Delta}{dy} + (1 - |K_{SS} K^{SS*}|^2) \Delta = 0, \quad (49)$$

where  $y \equiv \ln(1 - \chi_1 z)$ . Consequently, the general form of the solution is:

$$\Delta(z) = \mathcal{A}_1 (1 - \chi_1 z)^{(1+|K_{SS} K^{SS*}|)} + \mathcal{A}_2 (1 - \chi_1 z)^{(1-|K_{SS} K^{SS*}|)}, \quad (50)$$

where  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are complex constants determined by the initial conditions of  $\Delta$ . The solution (50) for  $\Delta(z)$  grows indefinitely as  $z \rightarrow -\infty$ , thus implying instability of the constant dilaton ( $S = S'$ ) solution. Therefore, on the  $z < 0$  side of extreme walls *the constant dilaton solution is always dynamically unstable*; any small deviation in the boundary condition  $S(z) = S'$  near  $z \sim 0$  leads to a space-time with a varying dilaton field where one encounters planar singularities, in general. Whether such singularities are naked or not depends on the nature of the self-dual Kähler potential.

On the  $z > 0$  side of the Type II wall the constant dilaton solution is also unstable;  $\Delta(z)$  satisfies the same type of equation as Eq. (49), but where now  $y \equiv \ln(1 + \chi_2 z)$  and  $\chi_2 = (e^{K/2} W)|_{S', T_2}$ . Thus, as  $z \rightarrow \infty$ ,  $|\Delta(z)| \rightarrow \infty$  and the solution is unstable. On the other hand, on the  $z > 0$  side of Type III walls  $\Delta(z)$  satisfies Eq. (49) with  $y \equiv \ln(1 - \chi_2 z)$ . In this case a general form of the solution is of the type:

$$\Delta(z) = \mathcal{B}_1 (1 - \chi_2 z)^{(1+|K_{SS} K^{SS*}|)} + \mathcal{B}_2 (1 - \chi_2 z)^{(1-|K_{SS} K^{SS*}|)}, \quad (51)$$

where  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are complex constants determined by the initial conditions of  $\Delta$ . Thus, for  $|K_{SS}| > K_{SS*}$ , the constant dilaton solution on the  $z > 0$  side of the Type III wall is unstable; however, for  $|K_{SS}| \leq K_{SS*}$ , the solution is *stable*. In the latter case the space-time reduces to the anti-de Sitter space-time where a finite value of  $z \sim 1/\chi_2$  corresponds to the time-like boundary of space-time.

In conclusion, in  $N = 1$  supergravity theory with a self-dual Kähler potential, extreme solutions corresponding to the constant dilaton,  $S = S'$  (where  $K_S|_{S'} = 0$ ), are always dynamically unstable (at least on one side of the wall).

## V. NON- AND ULTRA-EXTREME SOLUTIONS

In this section we shall analyse the non-extreme and ultra-extreme solutions. These are solutions that are not supersymmetric. They correspond to the domain wall backgrounds with moving wall boundaries. Unlike extreme solutions, which have supersymmetric embeddings and where solutions can be given for any thickness of the wall, we have only obtained non- and ultra-extreme solutions in the thin wall approximation. We thus employ the formalism spelled out in Section II.

In the following subsections we shall address the non-extreme solutions for the exponential dilaton potential, as well as the case with the self-dual dilaton potential. We shall also add a mass term for the dilaton field. Since we have found only numerical solutions for non-extreme walls, we shall confine the analysis to the non-extreme Type I walls (with

Minkowski space–time on one side of the wall) and reflection symmetric solutions, because only in this case the boundary conditions on either side of the wall can be specified uniquely. In subsection V A the boundary conditions for the non- and ultra-extreme Type I walls are written down explicitly, and the field equations are reduced to a first-order system. The results of the numerical integrations are presented and discussed. Subsection V B contains solutions for the reflection-symmetric cases, and in Subsection V C the effects of dilaton self-interactions are studied. Finally, in Subsection V D it is pointed out that the singularity-free self-dual dilatonic domain walls are dynamically unstable.

### A. Walls with Minkowski space–time on one side

We now consider the case where the dilaton potential outside the wall region has the form specified in Eq. (4) with  $f = e^{2\sqrt{\alpha}\phi}$  and  $\widehat{V}(\phi) = 0$ . Let the wall be non- or ultra-extreme with a non-vanishing  $V_0$  and a running dilaton on one side ( $z < 0$ ) and a Minkowski space with  $V_0 = 0$  and a constant dilaton on the other side ( $z > 0$ ). According to the results of Section II B, the boundary conditions are

$$\begin{aligned} \phi'|_{0^-} &= -\frac{1}{2}\sqrt{\alpha}\sigma, & a'|_{0^-} &= \frac{1}{2}\sigma - \beta \\ \phi'|_{0^+} &= 0 & a'|_{0^+} &= -\beta. \end{aligned} \quad (52)$$

$\beta < 0$  and  $\beta > 0$  represent an ultra-extreme and a non-extreme wall, respectively. Without loss of generality we have also chosen  $\phi|_0 = 0$  and normalized the metric coefficient  $a|_0 = 0$ . The choice  $\phi|_0 = \phi_0 \neq 0$  would correspond to the rescaling  $V_0 \rightarrow e^{2\sqrt{\alpha}\phi_0}V_0$ .

At the boundary  $z = 0^-$ , Eq. (9c) gives

$$V_0 - 3\beta\sigma + (3 - \alpha)\left(\frac{\sigma}{2}\right)^2 = 0. \quad (53)$$

For  $\alpha \neq 3$  the above equation reduces to

$$\sigma = \frac{2}{3 - \alpha} \left[ \sqrt{9\beta^2 + (3 - \alpha)^2\chi_1^2} + 3\beta \right], \quad (54)$$

where we have used the fact that  $\chi_1 = \sqrt{-V_0(\tau_1)/(3 - \alpha)}$ , as found in Eq. (39). In the case  $\alpha = 0$ ,  $\beta \neq 0$ , and  $\chi_1 = \sqrt{|V_0|/3}$ , one recovers the result from the non- and ultra-extreme anti-de Sitter–Minkowski space–time walls without a dilaton, and if  $\alpha = 0$ ,  $\beta > 0$ , and  $\chi_1 = 0$ , one recovers the dilaton-free non-extreme Minkowski–Minkowski walls [14,3].

For  $\alpha = 3$ , one finds

$$V_0 = 3\beta\sigma, \quad (55)$$

which indicates that in the non-supersymmetric case with  $\alpha = 3$ , the potential itself gets a non-zero value.

In order to integrate the field equations numerically we define a new  $\tilde{z}$ -coordinate by

$$\tilde{z} \equiv \chi_1 z. \quad (56)$$

Let us also define

$$\begin{aligned} B &\equiv a', & P &\equiv \phi'. \\ \tilde{\sigma} &\equiv \chi_1^{-1}\sigma, & \tilde{\beta} &\equiv \chi_1^{-1}\beta, \end{aligned} \quad (57)$$

where a prime now stands for a derivative with respect to the dimensionless coordinate  $\tilde{z}$ . Here,  $\sigma$  is as given in Eq. (54). Then the field equations (9) can be written as the first-order system

$$\begin{aligned} a' &= B, & B' &= e^{2a+2\sqrt{\alpha}\phi}(3-\alpha)/3 - 2P^2/3, \\ \phi' &= P, & P' &= -2BP - \sqrt{\alpha}e^{2a+2\sqrt{\alpha}\phi}(3-\alpha). \end{aligned} \quad (58)$$

The domain wall boundary conditions (52) then imply the following initial conditions for this dynamical system

$$\begin{aligned} a|_{0^-} &= 0, & B|_{0^-} &= \tfrac{1}{2}\tilde{\sigma} - \tilde{\beta}, \\ \phi|_{0^-} &= 0, & P|_{0^-} &= -\tfrac{1}{2}\sqrt{\alpha}\tilde{\sigma}. \end{aligned} \quad (59)$$

The conformal factor goes to zero faster than in the extreme space-times *both* in the non- and ultra-extreme cases (See Figs. 3 and 4). This is the case for *all* values of  $\alpha$  as long as  $\beta \neq 0$ . As illustrated in Fig. 3, when  $|\beta|$  is increased, the conformal factor decreases even faster. *Such domain walls thus always exhibit naked singularities.*

In order to understand these surprising results, it is instructive to look at the evolution of the dilaton. The general behaviour is illustrated in Fig. 5. The extreme solutions with  $0 < \alpha < 3$  are characterized by a delicate balancing of the kinetic and potential energies. As soon as the supersymmetry is broken, *i.e.*  $\beta \neq 0$ , the dilaton speeds away up (non-extreme case) or down (ultra-extreme case) its potential. As seen in Fig. 6, the kinetic energy eventually becomes dominant in both cases and is thus responsible for the appearance of a naked singularity.

*The string frame and its generalizations:* We would also like to comment on the physical significance of choosing a different frame for description of the space-time in the domain wall background. For the extreme domain walls there is a choice of a conformal frame defined by

$$\check{g}_{\mu\nu} \equiv e^{2\phi/\sqrt{\alpha}} g_{\mu\nu} \quad (60)$$

in which the space-time metric is *flat* [20], *i.e.*  $\check{g}_{\mu\nu} = \eta_{\mu\nu}$ . For  $\alpha = 1$  this frame is known as the “string frame”, *i.e.* the frame in which all the modes of the string theory couple coherently to the dilaton field. Since the metric is flat in the string frame, strings, which include all the modes, are “blind” to the curvature and to the singularities in the extreme domain wall backgrounds. One could then argue that the singularities are artefacts coming from the use of the Einstein frame, and that the fundamental physics as described using the metric  $\check{g}_{\mu\nu} = \eta_{\mu\nu}$ , *i.e.* the string frame metric and its generalizations, is well behaved.

In the extreme case there is always a frame with  $\check{g}_{\mu\nu} = \eta_{\mu\nu}$ , and in which (naked) singularities are swept under the rug. This, however, is not possible in the non- and ultra-extreme cases. In the non-extreme case the conformal factor in the  $\check{g}_{\mu\nu}$ -frame grows as one goes away from the wall, and in the ultra-extreme case it decreases, see Fig. 7. This implies that singularities are now felt in the  $\check{g}_{\mu\nu}$ -frame as well.

Furthermore, this result seems to imply a paradox. According to measurements in the Einstein frame, the area of an embedding surface at constant co-moving time decreases as one goes away from the wall on the side with the lowest value of the matter field potential. This is true in both the non- and ultra-extreme cases. This side is therefore *inside* the bubble. But now, according to the same type of measurements in the  $\check{g}_{\mu\nu}$ -frame, the side with the lowest value of the matter field potential is on the *outside* of the bubble in the ultra-extreme case. How can the same side be both on the inside and on the outside of the wall? The paradox is resolved when one realizes that this local definition of inside and outside depends on the conformal frame and that, because of the singularity cutting through space-time, there is no topological obstruction to turn it “inside out.”

## B. Reflection-symmetric walls

Now we consider a reflection-symmetric non-extreme wall (a special case of Type II walls) with non-vanishing  $V_0$  and a running dilaton.<sup>11</sup> The potential is taken to be that of Eq. (2) with  $f(\phi) = e^{2\sqrt{\alpha}\phi}$  and  $\hat{V}(\phi) = 0$ . According to Section II B, the boundary conditions are

$$\begin{aligned}\phi'|_{o-} &= -\phi'|_{o+} = -\frac{1}{4}\sqrt{\alpha}\sigma, \\ a'|_{o-} &= -a'|_{o+} = \frac{1}{4}\sigma.\end{aligned}\tag{61}$$

With these boundary conditions, Eq. (9c) gives

$$-3\beta^2 + V_0 + (3 - \alpha)\left(\frac{\sigma}{4}\right)^2 = 0.\tag{62}$$

If  $\alpha \neq 3$ , we find by use of Eq. (39) that

$$\sigma = 4\sqrt{\chi_1^2 + 3(3 - \alpha)^{-1}\beta^2}.\tag{63}$$

For  $\alpha = 0$ , this expression reduces to the one found for reflection-symmetric vacuum domain walls [3]. If  $\alpha = 3$ , then  $\sigma$  remains undetermined from this expression, but

$$V_0 = 3\beta^2.\tag{64}$$

This result again indicates that in the non-supersymmetric case with  $\alpha = 3$ , the potential itself is modified to be non-zero. Thus, these walls are different from reflection-symmetric domain walls in the background of a Zel'dovich fluid [36] or, which is equivalent, domain walls minimally coupled to a scalar field with no effective potential [37]. Yet, in all these cases one encounters naked singularities.

The boundary conditions for reflection-symmetric non-extreme walls are different from those of the wall adjacent to Minkowski space. In this case the singularity is further away from the wall (compare Figs. 3 and 8); however, as for the Type I non-extreme walls, the reflection-symmetric solutions always have naked singularities as well.

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<sup>11</sup>Of course, there are no reflection-symmetric ultra-extreme walls.

### C. Non- and ultra-extreme domain walls with a self-interacting dilaton

Let us now consider the situation where, supersymmetry breaking due to non-perturbative effects, introduces an additional self-interaction term in the dilaton potential. In general such a term is of a complicated form. One might hope that a dilaton self-interaction term, providing a mass for the dilaton, could stabilize the system and keep the dilaton from running away. For the sake of simplicity we consider an additional self-interaction potential  $\hat{V}$  of the form

$$\hat{V} = \lambda^2 \chi_1^2 \sinh^2 \omega \phi, \quad (65)$$

where  $\lambda$  and  $\omega$  are real constants. Note that in the cosmological picture, this potential has the opposite sign.

Since  $V(0) = V'(0) = 0$ , the boundary conditions for the equations of motion at  $z = 0^-$  for a wall adjacent to Minkowski space, remain as in Eq. (59).

The physics of this problem is most easily understood in the cosmological picture. We start by sending the dilaton up the exponential potential. In the non-extreme case it continues to roll up the potential until a singularity is reached. Adding a mass term, which would be negative in the cosmological picture, makes the dilaton decelerate less and as a result it rolls faster in the same direction. Hence, it is clear that such a self-interacting potential only contributes to an earlier appearance of the naked singularity in the non-extreme case. Examples are depicted in Fig. 9.

Now, in the ultra-extreme case, the dilaton reaches a maximum in its exponential potential and then returns. A self-interaction potential would tend to accelerate the dilaton in the positive direction. If this force is strong enough, the result will be as in the non-extreme case. If the exponential potential dominates, the dilaton will again return. In either case, the dilaton runs off and produces a naked singularity. In the ultra-extreme case, this can only be avoided by a fine-tuned self-interaction potential whose fine-tuning would have to depend also on  $\beta$ . Examples of these two scenarios are shown in Fig. 10.

The same qualitative features take place in the reflection-symmetric case as well.

### D. Self-dual dilaton coupling

Let us now assume that the dilaton coupling  $f(\phi)$  is self-dual. Namely, for a finite  $\phi = \phi'$ ,  $f(\phi)$  satisfies:

$$\left. \frac{\partial f}{\partial \phi} \right|_{\phi'} = 0 \quad (66)$$

and that  $\hat{V}(\phi) = 0$  (see Eq. (4)).

Note that the equation for the dilaton is of the form (see Eq. (9b))

$$2\phi'' + 4a'\phi' + e^{2a} \frac{\partial f(\phi)}{\partial \phi} V_0 = 0, \quad (67)$$

with  $\phi' = d\phi/dz$ . With the boundary conditions  $\phi|_0 = \phi'$  and  $\phi'|_0 = 0$ , the solution of the field equations corresponds to a dilaton frozen at  $\phi = \phi'$ . The global space-times of these solutions are then identical to those of the vacuum domain walls [3,6].



We would like to address a stability of the constant dilaton  $\phi(z) = \phi'$  solutions for extreme, non- and ultra-extreme solutions. In order to check the stability of such solutions, one adds to  $\phi(0) = \phi'$  an infinitesimal virtual displacement  $\delta(0)$ . Consequently,  $\delta(z)$  satisfies the following equation:

$$2\delta'' + 4a'\delta' + e^{2a} \left. \frac{\partial^2 f}{\partial \phi^2} \right|_{\phi'} V_0 \delta = 0 \quad (68)$$

which was obtained by expanding  $\phi(z) = \phi' + \delta(z)$  in Eq. (67) and using Eq. (66).

In Subsection IV C we have already shown that a supersymmetric embedding of an extreme solution with a constant (complex) dilaton  $S = S'$  *always* renders it unstable. We can also show an instability of such solutions by solving Eq. (68) directly, *i.e.* without reference to the effective dilaton potential restricted by  $N = 1$  supergravity theory.<sup>12</sup> On the  $z < 0$  side of the wall the constant dilaton ( $\phi = \phi'$ ) solution corresponds to the anti-de Sitter space-time. Thus,  $a(z)$  is a solution of Eq. (9a) with  $\beta = 0$  and  $\phi = \phi'$ . It is of the form:

$$a = -\ln(1 - \chi_1 z), \quad (69)$$

where  $\chi_1^2 = -f(\phi')V_0(\tau_1)/3$ . Eq. (67) is then of the form:

$$\frac{d^2 \delta}{dy^2} - 3 \frac{d\delta}{dy} - \frac{9}{4} b_0 \delta = 0, \quad (70)$$

where  $b_0 \equiv 2/3(\partial^2 f / \partial \phi^2) f^{-1}|_{\phi'}$  and  $y = \ln(1 - \chi_1 z)$ . The general form of the solution is

$$\delta(z) = C_1(1 - \chi_1 z)^{\lambda_1} + C_2(1 - \chi_1 z)^{\lambda_2} \quad (71)$$

with  $\lambda_{1,2} = 3(1 \pm \sqrt{1 + b_0})/2$ ;  $C_{1,2}$  are constants determined by initial conditions for  $\delta(0)$ . Consequently, as  $z \rightarrow -\infty$ ,  $|\delta| \rightarrow \infty$  in both cases, when the self-dual point corresponds to the maximum ( $b_0 > 0$ ) as well as the minimum of ( $b_0 < 0$ ) the effective dilaton potential. Thus, the extreme self-dual solutions with a frozen dilaton are always dynamically unstable.

We would now like to turn to the stability of the non-extreme solutions ( $\beta \neq 0$ ). On the  $z < 0$  side the space-time is anti-de Sitter with  $a(z)$  (a solution of Eq. (9a) with  $\phi = \phi'$  and  $\beta \neq 0$ ) of the form [3]:

$$e^{2a(z)} = \left( \frac{\beta}{\chi_1} \right) \sinh^{-2}[\beta(z - z')] \quad (72)$$

with  $e^{2\beta z'} \equiv 1 + 2(\beta/\chi_1)^2 + \beta/\chi_1 \sqrt{1 + (\beta/\chi_1)^2}$  and  $\chi_1^2 = -f(\phi')V_0(\tau_1)/3$ . Eq. (68) for  $\delta(z)$  can then be cast in the form:

$$\delta'' - 2\beta \coth[\beta(z - z')] \delta' - b_0 \beta^2 \sinh^{-2}[\beta(z - z')] \delta = 0, \quad (73)$$

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<sup>12</sup>The extreme solutions with a real dilaton field  $\phi$  can be viewed as corresponding to a special supersymmetric embedding, which renders the imaginary part of the complex field  $S$  constant.

where now  $b_0 \equiv (\partial^2 f / \partial \phi^2) f^{-1}|_{\phi'}$ . The general solution for  $\delta$  is complicated; however, as  $z \rightarrow -\infty$ , Eq. (73) reduces to:

$$w \frac{d^2 \delta}{dw^2} + 3 \frac{d\delta}{dw} - 4b_0 w \delta = 0, \quad (74)$$

where  $w = e^{\beta z}$ . Hence, as  $z \rightarrow -\infty$ ,  $\delta$  approaches a constant value, and thus the solution is *stable* under this perturbation.

Note that the same result applies to the non-extreme solutions where the constant dilaton solution corresponds to the Minkowski space-time, *i.e.*  $f(\phi')V_0(\tau_1) = 0$ . In this case

$$a(z) = \beta z \quad (75)$$

and Eq. (68) is again of the form (74) with  $w = e^{\beta z}$  but now  $b_0 \equiv -1/4\beta^{-2}(\partial^2 f / \partial \phi^2)|_{\phi'} V_0(\tau_1)$ . Clearly, as  $z \rightarrow -\infty$ ,  $\delta$  approaches a constant value, and therefore the solution is again stable.

Along similar lines one can prove that constant dilaton ultra-extreme solutions are also stable against an infinitesimal perturbation. In particular, in the case of Minkowski constant dilaton vacuum on the  $z > 0$  side of the wall,  $a(z) = \beta z$  and Eq. (68) is of the form (74) with  $w = e^{\beta z}$  and  $b_0 \equiv -1/4\beta^{-2}(\partial^2 f / \partial \phi^2)|_{\phi'} V_0(\tau_2)$ . The solution is of the form:

$$\delta(z) = \mathcal{D} Y^{-1} \mathcal{J}_1(Y), \quad (76)$$

where  $\mathcal{J}_1(Y)$  is the Bessel function of integer order one,  $Y = \sqrt{-4b_0} e^{\beta z}$  and  $\mathcal{D}$  is a constant. As  $z \rightarrow \infty$ ,  $Y \rightarrow \infty$  and  $\delta(z) \propto Y^{-3/2} \cos(Y - 3/4\pi) \rightarrow 0$ . Thus, the ultra-extreme solution is stable as well.

In conclusion, in a theory with a self-dual dilaton coupling extreme solutions with a constant dilaton are always unstable against small perturbation. The origin of the instability of extreme solutions may be related to an infinite extent of such planar configurations. On the other hand, non- as well as ultra-extreme solutions with a constant dilaton are always dynamically stable.

## VI. CONCLUSION

We have analysed the gravitational fields induced by dilatonic domain walls with an exponential dilaton coupling and with a self-dual dilaton coupling.

We have shown that generic non- and ultra-extreme dilatonic domain walls with an exponential dilaton coupling ( $e^{2\sqrt{\alpha}\phi}$ ) have naked singularities. In contrast, extreme walls have planar naked singularities for  $\alpha > 1$ , while for  $\alpha \leq 1$  the singularities are null. In the frame where the extreme domain wall metric is flat (and which for  $\alpha = 1$  corresponds to the string frame), the non-extreme domain walls still have naked singularities, whereas the ultra-extreme singularities are hidden behind horizons in this frame.

The ultra-extreme domain walls correspond to false vacuum decay bubbles and as such they might have a direct physical relevance if the Minkowski vacuum turns out to be unstable. A vacuum decay would, according to the theory we have used, force the dilaton to start running even if it had previously been trapped by a mass term or another self-interaction potential. The kinetic energy of the dilaton would then grow without bound and lead to a naked singularity.

For exponential dilaton coupling the non- and ultra-extreme domain walls have naked singularities, which cannot be avoided by adding a mass term for the dilaton. On the other hand, when the dilaton coupling is self-dual, there are singularity-free domain wall configurations: outside the domain wall region, the dilaton is then trapped at the self-dual point and the domain wall geometries reduce to those of the singularity-free vacuum domain walls. This class of solutions is dynamically stable for non- and ultra-extreme walls, but the extreme solutions of such a type are not dynamically stable.

We have therefore arrived at the conclusion that for a large class of theories with dilaton(s), the space-times of non- and ultra-extreme domain walls (as well as some extreme domain walls) are necessarily plagued with naked singularities. If one believes that the gravitational field of vacuum decay bubbles and non-extreme domain walls should be singularity-free, this observation imposes serious constraints on the phenomenological viability of such theories. In particular, the exponential dilaton coupling of the effective tree level action from superstrings is ruled out. The results obtained indicate that non-perturbative effects, *e.g.* non-perturbatively induced dilaton superpotential within supergravity theories and/or a non-perturbatively induced self-dual dilaton Kähler potential, should play a crucial rôle in altering qualitatively the space-time structure of the dilatonic domain wall space-times.

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# FIGURES

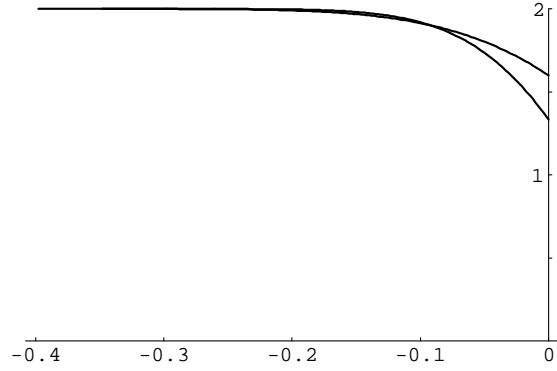


FIG. 1.  $\gamma$  versus  $z$  in units of  $\chi$  for extreme solutions with  $\alpha = 4$  (upper curve) and  $\alpha = 6$ . For  $0 \leq \alpha \leq 3$ , the equations of state are straight lines  $\gamma = 2\alpha/3$ .

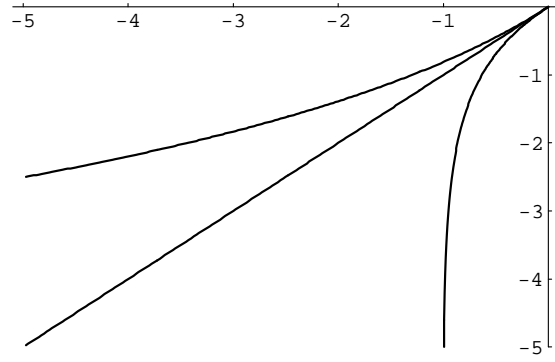


FIG. 2.  $a$  versus  $z$  in units of  $\chi$  with for extreme solutions with  $\alpha = 0.5$ ,  $\alpha = 1$ , and  $\alpha = 2$ , respectively. Solutions with  $\alpha > 1$  collapse to a naked singularity at a finite value of  $z$ .

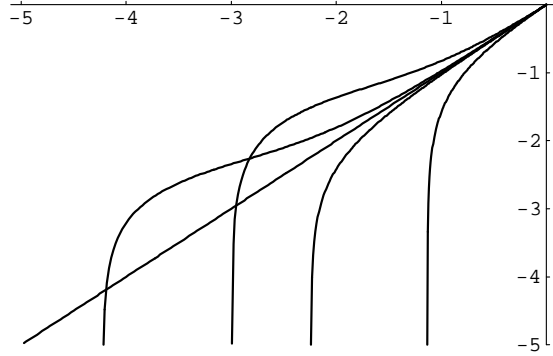


FIG. 3.  $a$  versus  $z$  in units of  $\chi$  with  $\alpha = 1$  for different values of  $\beta$ . Starting from the left at the bottom of the figure where  $a = -5$ , the curves correspond to  $\beta = 0$ ,  $\beta = -0.01$ ,  $\beta = -0.1$ ,  $\beta = 0.01$ , and  $\beta = 0.1$ .

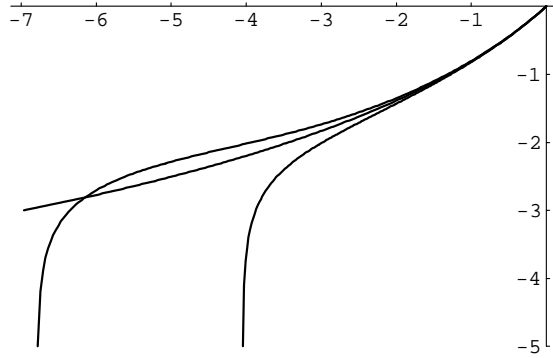


FIG. 4.  $a$  versus  $z$  in units of  $\chi$  with  $\alpha = 1/2$ . The curve starting in the middle corresponds to the extreme solution. The non-extreme case with  $\beta = 0.01$  becomes singular shortly after  $z = -4$ . The third curve corresponds to the ultra-extreme case with  $\beta = -0.01$ . It also ends in a singularity.

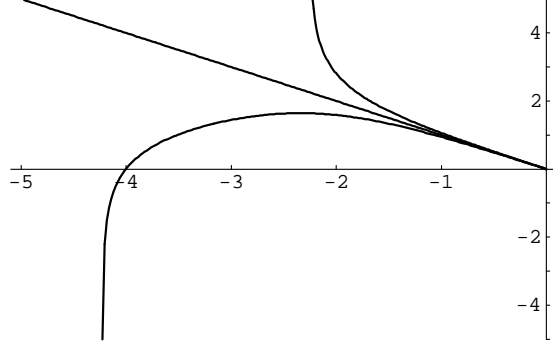


FIG. 5.  $\phi$  versus  $z$  in units of  $\chi$  with  $\alpha = 1$ . The straight line in the middle corresponds to the extreme solution. In the non-extreme case with  $\beta = 0.01$ , the dilaton grows without bound shortly after  $z = -2$ . For  $\beta = -0.01$ , corresponding to the ultra-extreme case, the dilaton has a turning point and then decreases without bound as the singularity is approached.

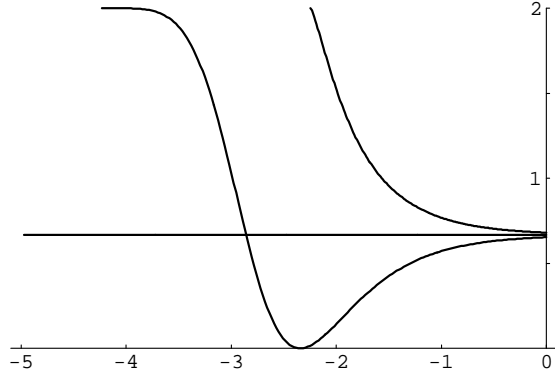


FIG. 6.  $\gamma$  versus  $z$  in units of  $\chi$  with  $\alpha = 1$ . The straight line corresponds to the extreme solution. In the non-extreme case with  $\beta = 0.01$ , the  $\gamma$  grows monotonically towards the limit  $\gamma = 2$  as the singularity is approached. The equation of state for  $\beta = -0.01$  drops down to  $\gamma = 0$  at the turning point of the scalar field, and then asymptotes towards  $\gamma = 2$  near the singularity.



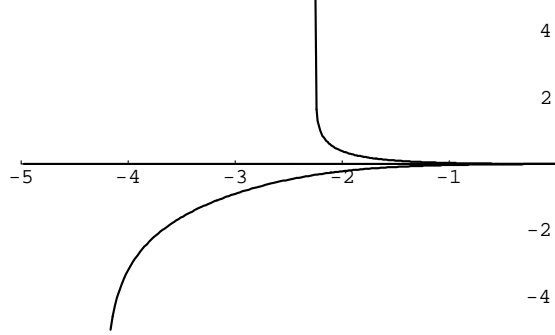


FIG. 7.  $\check{a}$  versus  $z$  in units of  $\chi$  with  $\alpha = 1$ . When  $\beta = 0$ ,  $\check{a} = 0$  and the metric is flat. In the ultra-extreme case  $\beta = -0.01$ ,  $\check{a} \rightarrow -\infty$  and the conformal factor goes to zero at finite  $z$ . In the non-extreme case with  $\beta = 0.01$ ,  $\check{a} \rightarrow \infty$ , and the conformal factor grows without bound at a finite value of  $z$ .

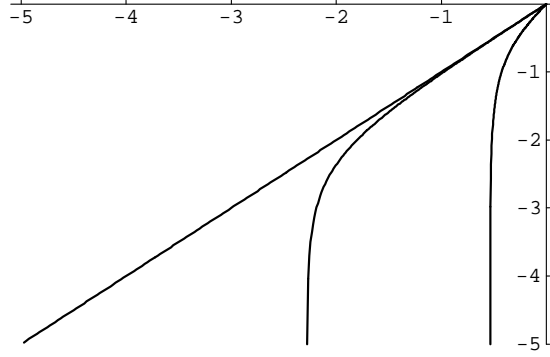


FIG. 8.  $a$  versus  $z$  in units of  $\chi$  for a reflection-symmetric wall with  $\alpha = 1$ . The straight line corresponds to the extreme solution. The other two curves represent non-extreme walls with  $\beta = 0.1$  and  $\beta = 1$ .

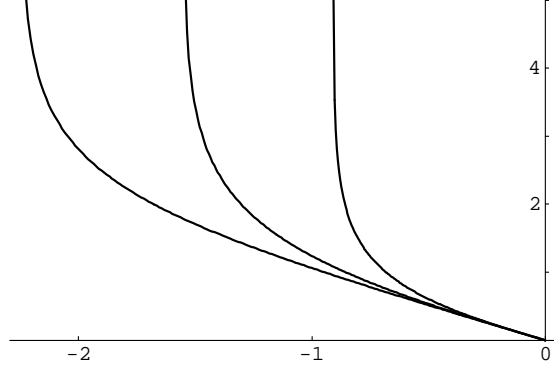


FIG. 9.  $\phi$  versus  $z$  in units of  $\chi$  with  $\alpha = 1$  in non-extreme solutions with  $\beta = 0.01$ . The left-most curve represents the non-extreme solution with no self-interaction term ( $\lambda = 0$ ). The other two correspond to  $\lambda = 1$  and  $\omega = 1$ , and  $\lambda = 1$  and  $\omega = 2$ , respectively.

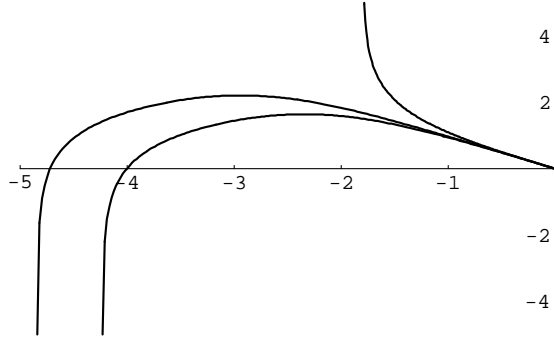


FIG. 10.  $\phi$  versus  $z$  in units of  $\chi$  with  $\alpha = 1$  in ultra-extreme solutions with  $\beta = -0.01$ . If the self-interaction is strong ( $\lambda = 1$  and  $\omega = 1$ ), the dilaton goes to positive infinity at the singularity. For a weaker self-interaction ( $\lambda = 1$  and  $\omega = 0.45$ ), the singularity comes later than in the case without self-interaction (lower-most curve).